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Jain states on a torus: an unifying description*

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Abstract: We analyze the modular properties of the effective CFT description for Jain plateaux, proposed in [1], corresponding to the fillings $\nu = \frac{m}{2pm+1}$. We construct its characters for the twisted and the untwisted sector and the diagonal partition function. We show that the degrees of freedom entering the partition function go to complete a Z_m-orbifold construction of the RCFT $\widehat{\mathrm{u}(1)} \times \widehat{\mathrm{su}(m)}_1$ proposed for the Jain states [2, 3]. The resulting extended algebra of the chiral primary fields can be also viewed as a RCFT extension of the $\widehat{\mathrm{u}(1)} \times \mathcal{W}_m$ minimal models [3]. For $m = 2$ we prove that our model, the TM, gives the RCFT closure of the extended minimal models $\widehat{\mathrm{u}(1)} \times \mathcal{W}_2$.

Keywords: Field Theories in Lower Dimensions, Conformal and W Symmetry.

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Contents

1. Introduction

Historically the motivations for the use of the Effective Conformal Field Theory (ECFT) in describing a Quantum Hall Fluid (QHF) at the plateau [4, 5] go back to the observation that the ground state wave function (Laughlin states) for the filling $\nu = 1/(2p+1)$ [6] can be written as correlators of vertex operators describing the anyon states (charged states), which were identified as primary states of a two dimensional chiral CFT with central charge $c = 1$ [7]. The ECFT description of a QHF at the plateau accounts for the incompressibility of the Laughlin fluid. The dynamical symmetry of this fluid in the disk geometry is the area-preserving diffeomorphisms of the plane which imply the $W_{1+\infty}$ algebra. This is the unique centrally extended quantum analogue of the classical area preserving diffeomorphisms algebra w_∞ on the circle. For general filling it is known that it is necessary to introduce neutral fields in order to describe the area-preserving edge deformations of the incompressible fluid. In this context two classes of CFTs have been proposed for Jain plateaux: the multi-components theory [2], characterized by the extended algebra $\widehat{\mathrm{u}(1)}\times\widehat{\mathrm{su}(m)}_1$ and the $W_{1+\infty}$ minimal models [8] both with central charge $c = m$. In spite of having the same spectrum of edge excitations, they manifest differences in the degeneracy of the states and in the quantum statistics.

An alternative and simple construction of the hierarchical model based upon the so called m-reduction procedure was given in [1], where a $c = m$ (daughter) CFT was obtained from a $c = 1$ (mother) one. The daughter theory preserves the $W_{1+\infty}$ symmetry of the mother theory. In fact it is well known that for any positive integer m, the $W_{1+\infty}$ algebra with central charge c is isomorphic to its subalgebra consisting of elements of degrees divisible by m and which appear with central charge mc [9, 10], so implying the incompressibility of the QHF for any given plateau.

This procedure was developed in [11] and employed for both the Jain fillings $\nu = \frac{m}{2pm+1}$ in [1] and the non standard ones $\nu = \frac{m}{pm+2}$ in [12] where it was shown how naturally it induces twists on the boundary, given by a Z_m group generated by the phases $\epsilon^j = e^{\frac{2\pi i j}{m}}$, $j = 1, \ldots, m$. As a consequence the chiral primary fields naturally appear as composite operators with a charged and a neutral component. It has been proved that their correlators reproduce the ground state wave function for the Jain as well as for the paired states, so generalizing the Laughlin type wave function on the plane [1, 12]. Only one $\widehat{\mathrm{u}(1)}$ charged current survives to the discrete twist group while the role of the neutral degrees of freedom is to cancel out the m-root singularity present in the correlators of the charged operators, so restoring locality.

In this paper we present a detailed analysis of such a model, the Twisted Model (TM), for the Jain states on a torus (genus $g = 1$ Riemann surface), so extending the previous description given on the plane [1]. For the paired states this analysis was given in [13].

We construct the characters associated to the extended chiral algebra and show that they provide a finite dimensional representation of the $\Gamma_0(2)$ subgroup of the modular group $\mathrm{SL}(2,\mathbb{Z})/Z_2$. Finally we exhibit the diagonal partition function which is invariant under such a subgroup. The resulting theory is the Z_m-orbifold of the Rational CFT (RCFT) describing the multi-components theory. Furthermore the extended algebra of the TM can be also viewed as a particular RCFT extension of the $W_{1+\infty}$ minimal models[1] [14].

Indeed their extensions are the constituents of the periodic-periodic (P-P) and periodic-antiperiodic (P-A) sector of this orbifold construction respectively. In addition to them the twisted (A-P) sector, is exactly what is needed in order to obtain the closure under

[1]We recall that the $W_{1+\infty}$ minimal models [14], which are defined by the full degenerate representations of $W_{1+\infty}$, are not a RCFT because it is impossible to build a finite extended algebra by assembling their primary fields, each taken with multiplicity one, that is closed under the S modular transformation. Instead the multi-components bosonic theories are RCFT extensions of the $W_{1+\infty}$ minimal models [8, 14, 15] where the extended primary fields are defined by collecting the fields of the full degenerate $W_{1+\infty}$ representations, but with multiplicity different from one. The TM is also a RCFT extension of the theory with a $W_{1+\infty}$ symmetry in which there are different multiplicities in any different sector.

$\Gamma_0(2)$. We should also comment on the presence of quasihole states in the twisted sector and clustering properties [1, 16] as it happens in the paired states.

All that emphasizes the key role of the m-reduction procedure in producing a RCFT which realizes a consistent $W_{1+\infty}$ extension of the classical area-preserving diffeomorphisms of the plane.

The paper is organized as follows:

In section 2 we review the K-matrix construction [2] of the multi-components theory of the Jain fillings and briefly summarize the properties of the ground state wave functions on the plane of a system of interacting layers which have been first analyzed in [17].

In section 3 we review the symmetries of the ECFT which are known in the literature: the RCFT with symmetry $\widehat{u(1)} \times \widehat{su(m)}_1$ and the $W_{1+\infty}$ minimal models [8]. We also give the decomposition of the $\widehat{u(1)}\times\widehat{su(m)}_1$ characters in terms of those of the $W_{1+\infty}$ minimal models [8, 14, 15]. Finally we indicate the Z_m discrete group with respect to which the orbifold construction is made.

In section 4 we briefly summarize the m-reduction construction on the plane (genus $g = 0$), which has been used in [1] to construct the TM for the Jain series at filling $\nu = \frac{m}{2pm+1}$.

In section 5 we generalize it to the torus topology (genus $g = 1$) and derive the extended chiral primary fields both for the twisted and the untwisted sectors and the corresponding characters. Then a comparison of the TM with the multi-components theory [2] is given and the features of the Z_m-orbifold construction shown. The factorization of the characters in terms of charged and neutral components is derived. Finally the special $m = 2$ case is also analyzed.

In section 6 we construct the complete diagonal partition function of the TM.

In section 7 we show that the TM provides a RCFT closure of the extended minimal models $\widehat{u(1)}\times\mathcal{W}_m\times Z_2$. In fact we prove that the P-A sector of the TM gives a realization of the chiral algebra containing only degenerate $W_{1+\infty}$ representations.

Finally in section 8 we give conclusions and comments.

In appendix A we describe the properties of the $\Gamma_0(2)$ subgroup of the modular group.

In appendix B we describe the extended odd RCFT $\widehat{u(1)}_q$ for the Laughlin filling, $\nu = 1/q$ where q is odd, which we use as mother theory for the m-reduction, pointing out the reason for requesting the closure under $\Gamma_0(2)$ only.

In appendix C we give the transformations of the characters of the TM under the elements of $\Gamma_0(2)$, showing that they provide a finite representation for it. We also prove that the TM diagonal partition function is invariant under its action.

2. K-matrix construction of the multi-components theory

The CFT for a m-levels QHF at filling $\nu = \frac{m}{2pm+1}$ was given in [2, 18, 19] introducing the symmetric and positive $\mathbf{K}$ matrix:

$$\mathbf{K} = \mathbf{1}_{m\times m} + 2p\mathbf{C}_{m\times m} \qquad \text{where: } \mathbf{C}_{m\times m} = \|1\|_{m\times m} . \tag{2.1}$$

The entries $\mathbf{K}_{(i,j)}$, depending on m and p, define the braiding factors between electrons in the i and j level. In this simplified picture the levels are considered independent and fully equivalent and we will refer to them also as layers. The wave function on the plane for such a system can be given in terms of $\mathbf{K}$ as:

$$f^{MC}(\{z_i^{(a)}\}) = \prod_{h=1}^{m}\prod_{i<j}\left(z_i^{(h)} - z_j^{(h)}\right)^{\mathbf{K}_{(h,h)}} \prod_{1=h<k}^{m}\prod_{i<j}\left(z_i^{(h)} - z_j^{(k)}\right)^{\mathbf{K}_{(h,k)}} e^{-\frac{1}{4}\sum_{h=1}^{m}\sum_i |z_i^{(h)}|^2} . \tag{2.2}$$

This is a special class of more general trial functions for the ground state of a multi-layer system. For the simplest case of only two layers (or levels) they can be codified in the so called (m,n,k) states introduced by Halperin [17]. The integers in the bracket are in correspondence with the entries of the $\mathbf{K}$ matrix, i.e. $m = \mathbf{K}_{(1,1)}$; $n = \mathbf{K}_{(2,2)}$ and $k = \mathbf{K}_{(1,2)}$. Some values of these integers give interesting models for special plateaux. The $(p+2,\, p+2,\, p)$ generalize the well studied (3,3,1) state and it was analyzed recently in the m-reduction approach in [13]. Another interesting case is the (1,1,1) for which theoretical studies predict the existence of a Josephson like effect. Nevertheless this state does not satisfy the non-degeneracy of the $\mathbf{K}$ matrix so it does not enter the theories we are studying now.

The CFT of the system with a general $\mathbf{K}$ matrix is realized by m free bosons with an extended chiral algebra. These fields satisfy the compactification conditions given by [2]:

$$\mathbf{Q}(z) = \mathbf{Q}(z) + 2\pi\mathbf{R}\mathbf{h} \tag{2.3}$$

where $\mathbf{h} \in \mathbb{Z}^m$ and $\mathbf{Q}(z)$ is the vector field with bosonic components $Q^{(i)}(z)$, $i = 1, \ldots, m$.

The $m{\times}m$ matrix $\mathbf{R}$ is defined by:[2]

$$\mathbf{R}^T\mathbf{R} = \mathbf{K} \tag{2.4}$$

and it is given explicitly by

$$\mathbf{R} = \mathbf{1}_{m\times m} + \frac{1}{m}\left(\sqrt{2pm+1} - 1\right)\mathbf{C}_{m\ \times m} . \tag{2.5}$$

It is actually a metric of the lattice Γ which refers to the electron field. The compactification matrix (2.3) for diagonal $\mathbf{h} = h\mathbf{t}$, where $\mathbf{t}$ is the charge vector which satisfies the condition $\mathbf{t}^T\left(\mathbf{K}\right)^{-1}\mathbf{t} = \nu$ and coincides with $\mathbf{t}^T = (1, \ldots, 1)$, defines the following compactification radius for the $Q^{(i)}$ bosons:

$$Q^{(i)}(z) = Q^{(i)}(z) + 2\pi r h \tag{2.6}$$

with $h \in \mathbb{Z}$, and r the odd compactification radius $r^2 = 2pm + 1$.

The primary fields for this theory are given by the vertex operators:

$$V^{\mathbf{a},\mathbf{b}}(z) =: e^{i\alpha_{\mathbf{a}}Q_{\mathbf{b}}(z)} : \qquad \text{with } (\mathbf{a},\mathbf{b}) \in \mathbb{Z}^m{\times}\mathbb{Z}^m \tag{2.7}$$

[2] $\mathbf{R}$ is the positive root of $\mathbf{K}$ and is well defined because $\mathbf{K}$ is a symmetric and positive definite matrix.

where $\alpha_{\mathbf{a}} = \mathbf{a}^T\mathbf{R}^{-1}$ and $Q_{\mathbf{b}}(z)$ is the boson with winding numbers $\mathbf{b}$; their conformal dimensions $h_{\mathbf{a},\mathbf{b}}$ are:

$$h_{\mathbf{a},\mathbf{b}} = \frac{1}{2}\left(\mathbf{a}^T\mathbf{R}^{-1} + \frac{1}{2}\mathbf{b}^T\mathbf{R}\right)\left(\mathbf{R}^{-1}\mathbf{a} + \frac{1}{2}\mathbf{R}\mathbf{b}\right). \tag{2.8}$$

The extended RCFT, the multi-components $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ theory, can be constructed by means of the following physical conditions [20]:

1. The topological order [19] given by $det\,(\mathbf{K})$ is an odd number and coincides with the number of primary fields of the extended algebra [20].

2. The characters of the extended theory are closed under $\Gamma_0(2)$.

3. The extended theory is given by the request that its conformal blocks correspond to primary fields with conformal dimensions which differ by a semi-integer from those of the lowest weight states.

The extended primary fields of the $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ theory are obtained by imposing that the weights are in the lattice $\mathbb{Z}_{\mathbf{K}} = Z^m/\mathbf{K}Z^m$ and the corresponding vertex operators are:

$$V_\lambda(z) =: e^{i\alpha_\lambda \cdot \mathbf{Q}(z)} : . \tag{2.9}$$

The corresponding characters are given in the standard form by:

$$\widetilde{\chi}_\lambda^{\mathbf{MC}}(w|\tau) = \mathrm{Tr}_{H_\lambda}\left(q^{\left(L_0 - \frac{m}{24}\right)} e^{2\pi i w J}\right) \tag{2.10}$$

where $L_0 = \frac{1}{2}\mathbf{n}^T\left(\mathbf{K}\right)^{-1}\mathbf{n}$ [19] is the eigenvalue of the L_0 mode of the Virasoro algebra and $J = \mathbf{t}^T\left(\mathbf{K}\right)^{-1}\mathbf{n}$ is the charge relative to $\mathbf{n}$, the vector defining the weight of the descendent state in the Hilbert space H_λ of the primary field with highest weight λ. Explicitly eq.(2.10) becomes:

$$\widetilde{\chi}_\lambda^{\mathbf{MC}}(w|\tau) = \frac{e^{-\pi m(2mp+1)\frac{(Imw)^2}{Im\tau}}}{\eta\left(\tau\right)^m} \sum_{\mathbf{a}\in Z^m} e^{2\pi i\left\{\frac{\tau}{2}\left[(\mathbf{K}\mathbf{a}+\lambda)^T(\mathbf{K})^{-1}(\mathbf{K}\mathbf{a}+\lambda)\right] + w\mathbf{t}^T\left[(\mathbf{K}\mathbf{a}+\lambda)\right]\right\}} \tag{2.11}$$

where the non-analytic phase was introduced in [3] and it is necessary to define properly the transport of charges. The extended theory $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ has $2pm+1$ independent primary states as it is requested by the topological order and the characters are closed under $\Gamma_0(2)$.

3. The symmetries of the ECFTs

In this section we briefly analyze the symmetry properties of the different ECFTs proposed to describe the Jain fillings.

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3.1 The $\widehat{\mathrm{u}(1)}\times\widehat{\mathrm{su}(m)}_1$ models

The factorization of the multi-components $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ model in a charged and a neutral sector allows us to evidence an $\widehat{\mathrm{u}(1)}_{m(2pm+1)}\times\widehat{\mathrm{su}(m)}_1$ algebraic structure. The charged sector has an $\widehat{\mathrm{u}(1)}_{m(2pm+1)}$ extended symmetry[3] while the neutral one has a full $\widehat{\mathrm{su}(m)}_1$ symmetry. Moreover the two sectors are not independent but are coupled by a discrete group so implying that a superselection rule, called m-ality rule, must be taken into account in order to build the physical excitations of the model (i.e. a holon-spinon topological coupling).

The factorization into a charged and neutral sector is explicitly obtained by diagonalizing the $\mathbf{K}$ matrix. The orthogonal matrix O_m, which does it, has been explicitly given in [21], and the new fields $\mathbf{Q}'(z) = O_m\mathbf{Q}(z)$ can be identified with those of ref. [1]. We define the charged boson $X(z) \equiv \frac{\sum_{l=1}^m Q_l'(z)}{\sqrt{m}}$ and the neutral vector $\Phi(z)$ with components $Q_i'(z)$ for $i = 1,\ldots,m-1$. The orthogonal matrix O_m can be written in terms of the base vectors $\mathbf{u}_i$ as follows:

$$O_m = \begin{pmatrix} \mathbf{u}_1 \ldots \mathbf{u}_m \\ \frac{1}{\sqrt{m}} \cdot\cdot \frac{1}{\sqrt{m}} \end{pmatrix}_{m\times m} \tag{3.1}$$

where the m-vectors $\mathbf{u}_i$ satisfy the constraint $\sum_i \mathbf{u}_i = 0$ and are related to the simple roots α_j and to the fundamental weights $\mathbf{\Lambda}_j$ of the finite algebra su(m) by the following relations:

$$\begin{aligned} \alpha_j &= \mathbf{u}_j - \mathbf{u}_{j+1}\,; & \mathbf{\Lambda}_l &= \sum_{j=1}^{l} \mathbf{u}_j \\ \mathbf{\Lambda}_l\cdot\alpha_j &= \delta_l^j\,; & \mathbf{\Lambda}_i\cdot\mathbf{\Lambda}_l &= i\left(1-\frac{l}{m}\right) \qquad \text{for } i \le l\,, \quad l = 1,\ldots,m-1\,. \end{aligned} \tag{3.2}$$

We put $\phi^j(z) = \mathbf{u}_j\cdot\mathbf{\Phi}(z)$ for $j = 1,\ldots,m-1$ so that the primary fields in the $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ theory factorize into a charged and a neutral part:

$$V_{\mathbf{X}}(z) = e^{i\alpha_X X(z)}\psi(z) \tag{3.3}$$

where $\alpha_X = \frac{\sum_{l=1}^m \lambda_l}{\sqrt{m(2pm+1)}}$, $\lambda \in \mathbb{Z}_{\mathbf{K}}$ and $\psi(z) = e^{i\sum_{j=1}^m \lambda_j \phi^j(z)}$.

The field $X(z)$ satisfies the following compactification condition, as a consequence of the compactification conditions for the $\mathbf{Q}(z)$ fields in eq. (2.6) :

$$X(z) = X(z) + 2\pi\, r_X l \qquad \text{where } r_X = \sqrt{m(2pm+1)} \tag{3.4}$$

Thus the charged component of $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ is just the $\widehat{\mathrm{u}(1)}_{m(2pm+1)}$ extended theory.

By using the neutral fields one gets the following currents:

$$J_{\alpha_{i,j}}(z) = c_{i,j} : e^{i(\phi^i(z)-\phi^j(z))} : \qquad \text{for } \alpha_{i,j} = \mathbf{u}_i - \mathbf{u}_j \tag{3.5}$$

$$J_h(z) = i\partial\Phi_h(z) \tag{3.6}$$

[3]It is defined in appendix B. It is an odd RCFT, that is a RCFT with respect to $\Gamma_0(2)$.

satisfying the $\widehat{\mathrm{su}(m)}_1$ algebra.[4] Let us recall that starting from $\mathbf{\Lambda}_j$, $j = 1, \ldots, m-1$, and the basic weight $\widehat{\mathbf{\Lambda}}_0$ of the affine algebra $\widehat{\mathrm{su}(m)}_1$, one can obtain all the other fundamental weights $\widehat{\mathbf{\Lambda}}_j = \mathbf{\Lambda}_j + \widehat{\mathbf{\Lambda}}_0$, $j = 1, \ldots, m-1$. For a general weight $\widehat{\mathbf{\Lambda}} = \sum_{i=0}^{m-1} n_i \widehat{\mathbf{\Lambda}}_i$ one can define the m-ality charge as $l = \sum_{i=1}^{m-1} i n_i$, $mod(m)$. It is a good quantum number useful to classify the excitations. As pointed out above, the theory $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ is not a simple tensor product of the charged and the neutral component. This is well evidenced by the explicit expression of the $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ characters given in eq. (2.11), which were introduced in [21] for the m=3 case and in [3] for generic m as:

$$\tilde{\chi}_b^{\mathbf{MC}}(w|\tau) = \sum_{l=0}^{m-1} \chi_l^{\widehat{\mathrm{su}(m)}_1}(\tau) \bar{K}_{(2pm+1)l+mb}^{[m(2pm+1)]}(w|\tau) \tag{3.7}$$

with $b = 0, \ldots, 2pm$ and where $\chi_l^{\widehat{\mathrm{su}(m)}_1}(\tau)$ are the characters of the $\widehat{\mathrm{su}(m)}_1$ representations with integrable highest weight (h.w.) $\widehat{\mathbf{\Lambda}}_l \in P_+^1$ (P_+^1 being the set of all the dominant weights of $\widehat{\mathrm{su}(m)}_1$) and conformal weight $h_l = \frac{l(m-l)}{2m}$ (i.e. a spinon with m-ality charge equal to l). Above $\bar{K}_c^{[m(2pm+1)]}(w|\tau)$ are the characters of the extended $\widehat{\mathrm{u}(1)}_{m(2mp+1)}$ theory and are explicitly given in appendix B.[5]

Furthermore the form of the $\mathbf{K}$ matrix allows to consider various discrete symmetries. Here we focus our attention on its symmetries under the permutations of the Landau levels. They can be analyzed in the diagonal basis in which the $\widehat{\mathrm{su}(m)}_1$ symmetry is evidenced. We distinguish two kinds of symmetries, one acting as an outer automorphism of the $\widehat{\mathrm{su}(m)}_1$ lattice and the other one associated to inner automorphisms of the algebra (i.e. the Weyl group). Inner automorphisms preserve the commutation relations of the algebra (although the action on the algebra generators is nontrivial). We let $D(g)$ and $D(\hat{g})$ stand for the symmetry group of the su(m) and $\widehat{\mathrm{su}(m)}$ Dinkin diagrams respectively. As it is well known the group of the outer automorphisms of $\hat{g}$, $O(\hat{g}) = D(\hat{g})/D(g)$, is the dual of the center of the group of g, i.e. $B(G)$ [22] which in the case of su(m) is Z_m. Moreover it acts on the finite part of $\hat{\Lambda}$ like an element w_g of the finite Weyl group (i.e. $g(\hat{\Lambda}) = (g-1)\hat{\Lambda}_0 + w_g \hat{\Lambda}$) by changing the m-ality of a fundamental representation: $g(l) = l+1$. The $B(G)$ group action on the algebra generators is trivial. We stress that outer automorphisms also must preserve the commutation relations of the algebra, so that the characters will be still given in terms of the $\widehat{\mathrm{su}(m)}_1$ ones. The extended $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ theory satisfies the important property that under the action of $O(\hat{g})$ all its extended characters are fixed points, that is they satisfy a gluing condition between the charged and the neutral sector which couples a Z_m subgroup of the $\widehat{\mathrm{u}(1)}$ to the m-ality charge in an invariant way. Thus the P-P, periodic-periodic sector of the TM will be invariant with respect to this coupled group (i.e. the permutation group

[4] $c_{i,j}$ are cocycle factors which allow to get the correct commutation relations.

[5] Notice that the parametrization given above is not unique due to the symmetry of the characters. They satisfy $\tilde{\chi}_{\lambda+(2mp+1)}^{\mathbf{MC}} = \tilde{\chi}_{\lambda}^{\mathbf{MC}}$ thus there are only $2mp+1$ independent primary fields. In [3] a parametrization which evidences the independence of the neutral component from the flux p was chosen. In this paper we give the same parametrization for the P-P and P-A sector. Nevertheless a different parametrization which is more useful for our approach will be given in the A-P sector. Obviously they are completely equivalent and they can be mapped one into the other.

acts as an inner automorphism for the full theory). This behavior is different from that of the paired states [13], where the Halperin theory does not coincide with the P-P untwisted sector of its Z_m-orbifold. The m-reduction procedure induces naturally this symmetry and gives the correct coupling between the charged and the neutral sector [1].

Furthermore quotienting by a discrete group that is not in the center it breaks all the affine representations; therefore the partition function of the TM in the twisted sector cannot be described in terms of the affine characters (i.e. the affine symmetry is broken in these sectors to the W_m one).

3.2 The W_m minimal model

The ECFT description of a QHF at the plateau is justified by the incompressibility of the Laughlin fluid. The dynamical symmetry of this fluid in the disk geometry is the area-preserving diffeomorphisms of the plane which imply the $W_{1+\infty}$ algebra. This is the unique centrally extended quantum analogue of the classical area preserving diffeomorphisms algebra w_∞ on the circle.

The infinite generators W_m^{n+1} of $W_{1+\infty}$ of conformal spin $(n+1)$ are characterized by a mode index $m \in Z$ and satisfy the algebra:

$$\left[W_m^{n+1}, W_{m'}^{n'+1}\right] = (n'm - nm')W_{m+m'}^{n+n'} + q(n,n',m,m')W_{m+m'}^{n+n'-2} \cdots + d(n,m)c\,\delta^{n,n'}\delta_{m+m'=0} \tag{3.8}$$

where the structure constants q and d are polynomials of their arguments, c is the central charge, and dots denote a finite number of similar terms involving the operators $W_{m+m'}^{n+n'-2l}$ [23, 25].

Such an algebra contains an abelian $\widehat{\mathrm{u}(1)}$ current for $n = 0$ and a Virasoro algebra for $n = 1$ with central charge c. Their zero modes eigenvalues are identified as the charge and angular-momentum of the edge excitations of the QHF.

In the literature [23, 24] the unitary representations of $W_{1+\infty}$ have integer central charge and can be of two types: generic or degenerate.

The generic representations, with central charge $c = m$, are equivalent to those of the multi-components bosons $\widehat{\mathrm{u}(1)}_K^{\otimes m}$, where the $\mathbf{K}$ matrices define the lattice $\mathbb{Z}_{\mathbf{K}} = \mathbb{Z}^m/\mathbf{K}\mathbb{Z}^m$ whose weights $\lambda = (\lambda_1, \ldots, \lambda_m)$ satisfy the conditions $\lambda_a - \lambda_b \notin \mathbb{Z}$, $\forall a \neq b \in 1, \ldots, m$.

The degenerate representations, with central charge $c = m$, are contained into those of the multi-components and satisfy the conditions $\lambda_a - \lambda_b \in \mathbb{Z}$.

In particular the full degenerate representations are those whose weights all define degenerate representations. These conditions select uniquely the lattices $\mathbb{Z}_{\mathbf{K}}$ as the ones generated by the $\mathbf{K}$ matrices of the Jain series. The theory with full degenerate representation only is also called $W_{1+\infty}$ minimal model [8].

The $W_{1+\infty}$ models are isomorphic to $\widehat{\mathrm{u}(1)} \times \mathcal{W}_m$ theory, where $\mathcal{W}_m$ is the algebra with central charge $c = (m-1)$, defined by the $q \to \infty$ limit of the Zamolodchikov-Fateev-Lukyanov algebra with $c = (m-1)\left(1 - \frac{n(n+1)}{q(q+1)}\right)$ [26]. In this limit an infinity of degenerate representations appears which are in one-to-one relation to the degenerate representations of the above minimal models.

The $\mathcal{W}_m$ algebra can be also defined by a coset construction of the kind $W(\hat{g}/g;k)$ based on the Casimir operators of a finite algebra g (see [23] and [14] for details). The relevant coset for Jain series is $W(\widehat{\mathrm{su}(m)}_k/\,\mathrm{su}(m);k=1)$, which involves the finite algebra $\mathrm{su}(m)$, thus the central charge of the $\mathcal{W}_m$ has the same value of the full theory $\widehat{\mathrm{su}(m)}_1$, i.e. $c_{\mathcal{W}_m}=c_{\widehat{\mathrm{su}(m)}_1}=m-1$. The h.w. representations of $\mathcal{W}_m$ are defined by decomposing those of $\widehat{\mathrm{su}(m)}_1$ in terms of $\mathrm{su}(m)$ ones. The characters of the coset $\mathcal{W}_m$ are thus the branching functions constructed by decomposing the characters of the affine algebra in terms of those of the finite one:

$$\chi^{\widehat{\mathrm{su}(m)}_1}_{\widehat{\mathbf{\Lambda}}_l}(\xi|\tau)=\sum_{\mathbf{\Lambda}:\widehat{\mathbf{\Lambda}}\in P^1_+\cap\Omega_{\widehat{\mathbf{\Lambda}}_l}} b^{\Lambda}_{\widehat{\mathbf{\Lambda}}_l}(\tau)\chi^{\mathrm{su}(m)}_{\mathbf{\Lambda}}(\xi) \tag{3.9}$$

where $\Omega_{\widehat{\mathbf{\Lambda}}_l}$ is the set of the weights in the h.w. representation of $\widehat{\mathrm{su}(m)}_1$ relative to $\widehat{\mathbf{\Lambda}}_l$, the integrable h.w., $\mathbf{\Lambda}$ is the finite part of the affine weight $\widehat{\mathbf{\Lambda}}$.

The above $\mathrm{su}(m)$ characters are explicitly given by:

$$\chi^{\mathrm{su}(m)}_{\mathbf{\Lambda}}(\xi)=\frac{\sum_{w\epsilon W}\epsilon(w)e^{w(\mathbf{\Lambda}+\rho)\cdot\xi}}{\sum_{w\epsilon W}\epsilon(w)e^{w(\rho)\cdot\xi}} \tag{3.10}$$

where $\mathbf{\Lambda}=\mathbf{\Lambda}_l+\gamma$ and $\gamma\in Q$, the set of roots of $\mathrm{su}(m)$, w is an element in the Weyl group W of the $\mathrm{su}(m)$ and $\epsilon(w)$ is its parity. Thus the characters of $\mathcal{W}_m$ are defined by $\chi^{\mathcal{W}_m}_{\mathbf{\Lambda}}(\tau)\equiv b^{\Lambda}_{\widehat{\mathbf{\Lambda}}_l}(\tau)$ and their explicit expressions are given by:

$$\chi^{\mathcal{W}_m}_{\mathbf{\Lambda}}(\tau)=\frac{q^{\frac{\Lambda^2}{2}}}{\eta(q)^{m-1}}\prod_{\alpha\epsilon\Delta_+}(1-q^{(\Lambda+\rho)\cdot\alpha})\,. \tag{3.11}$$

For $z\to 0$ the characters $\chi^{\mathrm{su}(m)}_{\mathbf{\Lambda}}(\xi=z\rho)$ go to the dimension $d_{\mathrm{su}(m)}(\mathbf{\Lambda})$ of the $\mathbf{\Lambda}$ representation of the finite algebra $\mathrm{su}(m)$. Thus the $\widehat{\mathrm{su}(m)}_1$ characters can be written in the form:

$$\chi^{\widehat{\mathrm{su}(m)}_1}_l(\tau)\equiv\lim_{z\to 0}\chi^{\widehat{\mathrm{su}(m)}_1}_{\widehat{\mathbf{\Lambda}}_l}(\xi=z\rho|\tau)=\sum_{\Lambda:\widehat{\Lambda}\in P^1_+\cap\Omega_{\widehat{\mathbf{\Lambda}}_l}} d_{\mathrm{su}(m)}(\mathbf{\Lambda})\chi^{\mathcal{W}_m}_{\mathbf{\Lambda}}(\tau)\,. \tag{3.12}$$

The equivalence $\widehat{\mathrm{su}(m)}_1=\mathrm{su}(m)\times\mathcal{W}_m$ gives $\widehat{\mathrm{su}(m)}_1$ as a RCFT extension of the algebra $\mathcal{W}_m$. Eq. (3.12) shows that any integrable h.w. $\widehat{\mathbf{\Lambda}}_l$ representation of $\widehat{\mathrm{su}(m)}_1$ is given by collecting infinite representations $\mathbf{\Lambda}$ of $\mathcal{W}_m$, such that $\widehat{\Lambda}\in P^1_+\cap\Omega_{\widehat{\mathbf{\Lambda}}_l}$, each one with multiplicity $d_{\mathrm{su}(m)}(\mathbf{\Lambda})$.

Nevertheless this extension is not a minimal one which should be given by collecting infinite primary fields of $\mathcal{W}_m$, everyone with multiplicity one.

The main point for this minimal $\mathcal{W}_m$ theory is that it is not a RCFT, that is the characters relative to the extended primary fields are not closed under the modular transformation S. We propose a theory in which both symmetries can be taken into account, being realized in different sectors of the full TM. It is based on another interesting realization of the $\mathcal{W}_m$ algebra given in [9] by using twisted bosons. The representation theory for this realization was studied in [10]. This realization is the relevant one for the m-reduction and introduces new topological sectors to the description of the Jain fillings. In order to obtain the twisted characters of the $\mathcal{W}_m$ algebra it is necessary to add the special vector $\frac{\rho}{m} = \frac{1}{m}\sum_{i=1}^{m-1}\mathbf{\Lambda}_i$, with $\rho^2 = \frac{m(m^2-1)}{12}$, to all the representations. Nevertheless these are not representations of the algebra $\widehat{\mathrm{su}(m)}_1$, so this symmetry is explicitly broken in this vacuum. This vector changes the conformal dimensions of the vacuum state according to:

$$h_{\mathbf{\Lambda}} = \frac{1}{2}(\mathbf{\Lambda} + \frac{\rho}{m})^2 = \frac{\mathbf{\Lambda}^2}{2} + \frac{m^2-1}{24m} + \frac{\tilde{f}}{2m} \tag{3.13}$$

so that all the $n_i = \delta_i^l$ are simply translated by $1/m$ with respect to those of the minimal models. Indeed, as it follows from the modular invariance of the theory, we find that the neutral characters for the twisted sector can be given in terms of $\chi_l^{\widehat{\mathrm{su}(m)}_1}(\rho\tau/m|\tau)$.

Moreover not only the $\widehat{\mathrm{u}(1)}\times\widehat{\mathrm{su}(m)}_1$ is contained in our RCFT and it coincides with the P-P sector of the TM but also an extended $\mathcal{W}_m$ minimal model is realized in the P-A sector, as it will be shown explicitly in the following sections.

4. The m-reduction on the plane

In this section we briefly review the m-reduction procedure on the plane as a starting point for its generalization to the torus which is the main content of this paper. Our approach is meant to describe all the plateaux with odd denominator starting from the Laughlin filling $\nu = 1/(2pm+1)$, which is described by a CFT with $c = 1$, in terms of a scalar chiral field compactified on a circle with radius $R^2 = 1/\nu = 2pm+1$ (or its dual $R^2 = 4/(2pm+1)$). Then the u(1) current is given by $J(z) = i\partial_z Q(z)$, where $Q(z)$ is the compactified Fubini field with the standard mode expansion:

$$Q(z) = q - i\,p\,lnz + \sum_{n\neq 0}\frac{a_n}{n}z^{-n} \tag{4.1}$$

with a_n, q and p satisfying the commutation relations $[a_n, a_{n'}] = n\delta_{n,n'}$ and $[q,p] = i$.

The primary fields are expressed in terms of the vertex operators $U^{\alpha_s}(z) =: e^{i\alpha_s Q(z)}:$ with $s = 1,\ldots,2pm+1$ and conformal dimension $h = \frac{s^2}{2(2pm+1)}$. The $h = pm + \frac{1}{2}$ field $(s+2pm+1)$ describes the electron with electric charge $q_e = 1$ in units of the electron charge e and magnetic one $q_m = 2pm+1$ in unit of hc/e, while to other $2pm$ primary fields correspond to anyons with lowest charges $q_e = \frac{1}{2pm+1}$ and $q_m = 1$. The dynamical symmetry is given by the $W_{1+\infty}$ algebra with $c = 1$, whose generators are simply given by a power of the current $J(z)$.

We start with the set of fields in the above CFT (mother theory). Using the m-reduction procedure, which consists in considering the subalgebra generated only by the modes in eq. (4.1), which are multiple of an integer m, we get the image of the twisted sector of a $c = m$ orbifold CFT (daughter theory, i.e. the TM) which describes the Lowest Landau Level (LLL) dynamics of the new filling $\nu = m/(2pm+1)$.

Then the fields in the mother CFT can be factorized into irreducible orbits of the discrete Z_m group which is a symmetry of the TM and can be organized into components which have well defined transformation properties under this group. To compare the orbifold so built with the $c = m$ CFT, we use the mapping $z \to z^{1/m}$and the isomorphism defined in ref. [11] between fields on the z plane and fields on the z^m covering plane given by the following identifications: $a_{nm+l} \longrightarrow \sqrt{m} a_{n+l/m}$, $q \longrightarrow \frac{1}{\sqrt{m}} q$.

We perform a "double"-reduction which consists in applying this technique into two steps.

1. The m-reduction is applied to the Fubini field $Q(z)$. This induces twisted boundary conditions on the currents. It is useful to define the invariant scalar field:

$$X(z) = \frac{1}{m} \sum_{j=1}^{m} Q(\varepsilon^j z) \tag{4.2}$$

   where $\varepsilon^j = e^{i\frac{2\pi j}{m}}$, corresponding to a compactified boson on a circle with radius now equal to $R_X^2 = R^2/m = 2p + 1/m$. This field describes the u(1) electrically charged component of the new filling.

   On the other hand the non-invariant fields defined by

$$\phi^j(z) = Q(\varepsilon^j z) - X(z)\,, \qquad \sum_{j=1}^{m} \phi^j(z) = 0 \tag{4.3}$$

   naturally satisfy twisted boundary conditions so that the $J(z)$ current of the mother theory decomposes into a charged current given by $J(z) = i\partial_z X(z)$ and $m-1$ neutral ones $\partial_z \phi^j(z)$ [1].

2. The m-reduction applied on the vertex operators $U^{\alpha_s}(z)$ of the mother theory also induces twisted boundary conditions on the vertex operators of the daughter CFT. The discrete group used in this case is just the m-ality group.

   The vertex operator in the mother theory can be factorized into a vertex that depends only on the invariant field:

$$\mathcal{U}^{\alpha_s}(z) = z^{\frac{\alpha_s^2(m-1)}{m}} : e^{i\alpha_s\, X(z)} : \tag{4.4}$$

   and in vertex operators depending on the $\phi^j(z)$ fields. We introduce the neutral component:

$$\psi_1(z) = \frac{z^{\frac{1-m}{m}}}{m} \sum_{j=1}^{m} \varepsilon^j : e^{i\phi^j(z)} : \tag{4.5}$$

   which invariant under the twist group given in 1) and has m-ality charge $l = 1$.

The set of primary fields generated by their product can be given in terms of the fundamental representations Λ_l of the su(m) Lie algebra. By looking at their conformal dimension we are led to identify them as $\widehat{\mathrm{su}(m)}_1$ spinons given in a m-ality diagonal basis. Indeed we can define the full set of fields:

$$\psi_l(z) = \sum_{j_1>j_2>\cdots>j_l} f(\varepsilon^{j_1}, \ldots, \varepsilon^{j_l}, z) : e^{i\phi^{j_1}(z)} \cdots e^{i\phi^{j_l}(z)} : \tag{4.6}$$

where the functions $f(\varepsilon^{j_1}, \ldots, \varepsilon^{j_l}, z)$ can be extracted from the OPE relations. The sum takes into account the fact that any field can be associated to the l-th fundamental representation of su(m) (namely, the antisymmetric tensor representation) and then have m -ality l.

Higher spin currents in $W_{1+\infty}$ algebra are given by the infinite generators in the enveloping algebra of the u(1) charged sector and by the $\mathcal{W}_m$ currents obtained from the neutral sector. The explicit form for $m \leq 4$ was given in [9]. We report here only the first element beyond the spin two of the series which in our basis is expressed as:

$$W^3(z) = \frac{1}{2\sqrt{m}} \sum_{j,j',j''=1}^{m} : \partial_z \phi^j(z) \partial_z \phi^{j'}(z) \partial_z \phi^{j''}(z) : . \tag{4.7}$$

All the fields given in eq. (4.6) are invariant under the Z_m group twisting the $\phi^j(z)$ fields. The remaining primary fields in the neutral sector which are not invariant are given by:

$$\bar{\psi}_g(z) = \frac{z^{\frac{(1-m)}{m}}}{m} \sum_{j=1}^{m} \varepsilon^{gj} : e^{i\phi^j(z)} : \tag{4.8}$$

with $g = 2, \ldots, m$. This decomposition breaks explicitly the finite su(m) symmetry of the multi-components model. As we will show in the next section the residual symmetry of the twist invariant sector is just the $W_{1+\infty}$ one.

5. The m-reduction on the torus

In the following we consider the case of prime m only, the more general case will be the subject of a forthcoming paper.

5.1 The TM for prime $m > 2$

The present discussion is valid for any prime $m > 2$ while in section (5.2) it is analyzed the special $m = 2$ case. This is not only to clarify the construction in the simplest case but also for the peculiarity of the $m = 2$ case in which the m-reduction furnishes special results.

The extended TM theory is the realization on the torus topology of that proposed in [1] to describe the plateaux for the Jain series $\nu = \frac{m}{2pm+1}$ on the plane. We can obtain the full theory by applying the m-reduction on the torus and by implementing the modular invariance condition. The mother theory describes the Laughlin filling $\nu = \frac{1}{2mp+1}$ corresponding to the $\widehat{\mathrm{u}(1)}_{2mp+1}$ symmetry (see appendix B for details).

By applying the m-reduction on the fields in the mother theory we obtain the twisted sector of the TM immediately. In this procedure it is necessary to take into account two points:

1. The $\widehat{\mathrm{u}(1)}_{2pm+1}$ mother theory is not an ordinary RCFT but a theory which is invariant only under $\Gamma_0(2)$.

2. The TM is a fermionic theory so it must be of the same type of the mother theory (i.e. the conformal dimensions of its primary fields are defined only $mod(1/2)$).

The extended primary fields in the twisted sector have conformal dimensions:[6]

$$h_{(s,\tilde{f})} = \frac{s^2}{2(2pm+1)m} + \frac{m^2-1}{24m} + \frac{\tilde{f}}{2m} \tag{5.1}$$

and are parametrized by $s = 0,\ldots,2pm$, which is the quantum number distinguishing the fields in the mother theory and by $\tilde{f} = 0,\ldots,m-1$, which indicates the twisted sector in the model. The term $\frac{m^2-1}{24m}$ is due to the twisted conditions. The condition 2) implies that the number of conformal blocks of the extended theory are m instead of $2m$. The corresponding characters can be easily given in terms of those of the mother theory:

$$\chi_{(s,\tilde{f})}(w|\tau) = \frac{1}{m}\sum_{j=0}^{m-1} e^{-\frac{2\pi i}{m}(2j)(\frac{\tilde{f}}{2}+\frac{s^2}{2(2pm+1)}-\frac{1}{24})}\bar{K}_s^{(2pm+1)}\left(w|\frac{\tau+2j}{m}\right). \tag{5.2}$$

The closure under $\Gamma_0(2)$ of the TM allows us to obtain part of the fields in the untwisted sector. We get $2pm+1$ new primary fields for this sector with conformal dimensions:

$$\tilde{h}_b = \frac{(mb)^2}{2(2pm+1)m}. \tag{5.3}$$

They can be identified with the m-particles introduced in [20] and are described by the characters

$$\tilde{\chi}_b(w|\tau) = \frac{1}{\sqrt{m}}\bar{K}_b^{(2pm+1)}(mw|m\tau). \tag{5.4}$$

Their peculiarity is that the neutral part of the vertex operator is the identity. It gives a contribution only by means of its descendents which are just the $\mathcal{W}_m$ currents.

Although these fields are sufficient to satisfy the $\Gamma_0(2)$ covariance they are not the complete set of fields in the TM. As for the paired states case analyzed in [12], the periodic-periodic sector of the theory can be obtained by exploring the contents of the orbifold theory $\widehat{\mathrm{u}(1)}_K^{\otimes m}/Z_m$, which is the daughter CFT we obtained by m-reduction. Therefore in the next section we study this orbifold in detail and extract the information we need to complete the model.

[6]We remember that the parametrization here used for the untwisted characters is different from that of the twisted one.

5.1.1 The orbifold $\widehat{\mathrm{u}(1)}_K^{\otimes m}/Z_m$

Starting with the $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ theory, which is invariant under the discrete symmetries analyzed in section (3.1), we perform an orbifold construction with respect to the discrete Z_m group. It should be noticed that the $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ is a $\Gamma_0(2)$-RCFT so the generated orbifold will be of the same kind.

All the primary fields of the extended $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ theory correspond to diagonal weights $\mathbf{b} = b\mathbf{t}$. Defining g as the generator of the Z_m group, permuting the levels, we see that all these fields are invariant under its action and are fixed points with respect to this symmetry. The stabilizer S_b for a general field corresponding to the weight $\mathbf{b}$ is just Z_m, i.e. $S_b = \{g^i :, \ i = 0, \ldots, m-1\}$ and it has m elements.

The untwisted P-A sector of the orbifold is obtained introducing a new primary field for any element $\pi \in S_b$ and any field in the mother theory. The generic conformal field is parametrized by the index $\mathbf{b}$ and an element $\pi \in S_b$ and its conformal dimension given by:

$$\tilde{h}_{(b,(\mathbf{1},\pi))} = \frac{(mb)^2}{2(2pm+1)m} \tag{5.5}$$

for $\pi \neq g^0$. The corresponding characters are

$$\tilde{\chi}_{(b,(\mathbf{1},\pi))}(w|\tau) = \begin{cases} \tilde{\chi}_b^{\mathbf{MC}}(w|\tau) & \text{for } \pi = g^0 = \mathbf{1} \\ \bar{K}_b^{(2pm+1)}(mw|m\tau) & \text{for } \pi = g^i\,, \qquad i = 1, \ldots, m-1\,. \end{cases} \tag{5.6}$$

Eq. (5.6) implies that the conformal block relative to the identity coincides with that of the $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ theory while there is a $m-1$ degeneracy in the other blocks. The characters $\chi_{(b,(\mathbf{1},g^i))}(w|\tau)$ of the orbifold are equivalent to the characters $\sqrt{m}\tilde{\chi}_b(w|\tau)$ given in eq. (5.4).

The twisted A-P sector of the $\widehat{\mathrm{u}(1)}_K^{\otimes m}/Z_m$ model can be obtained by the request of covariance under $\Gamma_0(2)$. Indeed from $\tilde{\chi}_{(s,(\mathbf{1},g^i))}(w|\tau)$ we can generate the characters of the twisted sector by means of the transformation S. The closure under T^2 implies that also the characters $\chi_{(s,(g^i,g^j))}(w|\tau) = \bar{K}_s^{(2pm+1)}(w|\frac{\tau+2j}{m})$, for $j = 1, \ldots, m-1$, belong to this sector.[7]

The full twisted sector of this $\widehat{\mathrm{u}(1)}_K^{\otimes m}/Z_m$ orbifold is given then by the characters

$$\chi_{(s,(g^i,g^j))}(w|\tau) = \bar{K}_s^{(2pm+1)}\left(w|\frac{\tau+2j}{m}\right) \tag{5.7}$$

for $s = 0, \ldots, 2pm$, $i = 1, \ldots, m-1$ and $j = 0, \ldots, m-1$. These characters are degenerate with respect to the index i as it was pointed out before.

As we have seen the P-P sector of this orbifold coincides with the $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ itself. The remaining part of the untwisted sector coincides with the untwisted sector of the theory built by the m-reduction procedure, when we take into account the $m-1$ degeneracy. The twisted A-P sector is given by characters of eq. (5.7). Looking at eq. (5.7) and eq. (5.2), we find the relationship between these characters and the characters obtained

[7]The closure is given when $2j \in 0, 2(m-1)$ because the characters $\bar{K}_\lambda^{(2pm+1)}(w|\tau)$, of $\widehat{\mathrm{u}(1)}_{2pm+1}$, are closed under T^2 and not under T.

by m-reduction. Indeed as a consequence of the invertibility of the equality (5.2) for any fixed j, the characters in eqs. (5.7) and (5.2) simply define two different basis of the same twisted sector.

Thus we can conclude that the m-reduction procedure generates the non periodic part of the untwisted sector (P-A) and the twisted sector (A-P) of the Z_m-orbifold of $\widehat{\mathrm{u}(1)}_K^{\otimes m}$, by considering that any extended primary field has degeneracy $m-1$.

5.1.2 Charged and neutral components of the characters of the TM

In ref. [1] the TM was given in terms of charged and neutral components on the plane. There it was shown that the charged component coincides with the compactified theory with $R^2 = (2mp+1)/m = 1/\nu$. In the torus topology this separation can be given by the decomposition of the characters in terms of those of the charged and neutral ones. This decomposition also tells us the way in which the primary fields of the charged and neutral components must combine in order to generate the extended primary fields of the full TM theory.

For the P-P sector the separation is given in eq. (3.7), while for the P-A sector of the TM we obtain

$$\tilde{\chi}_b(w|\tau) = \tilde{N}(\tau)\bar{K}_{mb}^{[m(2pm+1)]}(w|\tau) \tag{5.8}$$

where $\tilde{N}(\tau) = \frac{1}{\sqrt{m}}\frac{\eta(\tau)}{\eta(m\tau)}$ represents the neutral contribution.

To clarify the relationship between the TM and the $\mathcal{W}_m$ minimal model we point out that the twist group acts only on the su(m), while the $\mathcal{W}_m$ algebra is unaffected under it. Therefore we can write $(\widehat{\mathrm{su}(m)}_1/Z_m = \mathrm{su}(m)/Z_m \times \mathcal{W}_m)$ and the effect of the twist is only in the reduction of the degeneracy of the $\mathcal{W}_m$ representations as it happens in the minimal models.

Nevertheless, while the characters of $\widehat{\mathrm{su}(m)}_1$ transform covariantly under the modular group, the $\mathcal{W}_m$ ones do not close under this group. That can be easily understood if one notes that the $\mathrm{SL}(2,\mathbb{Z})$ also acts on the representations of the finite su(m) algebra. Therefore one cannot have a covariance of the $\mathcal{W}_m$ characters without a covariance of the degenerate vacuum states of dimension $d(\mathbf{\Lambda}_a + \gamma)$. To do that we relate a finite subgroup of the modular group to the su(m) automorphisms.

By using the identity

$$\frac{\eta(\tau)^m}{\eta(m\tau)} = \frac{\prod_{j=1}^m \Theta_3\left(w - \frac{j}{m}|\tau\right)}{\Theta_3\left(m(w - \frac{m+1}{2})|m\tau\right)} \tag{5.9}$$

we give the neutral sector in terms of the characters of $\widehat{\mathrm{su}(m)}_1$:[8]

$$\tilde{N}(\tau) = \frac{1}{\sqrt{m}}\chi_0^{\widehat{\mathrm{su}(m)}_1}\left(\frac{\rho}{m}|\tau\right)$$

where ρ is the Weyl vector of the algebra. In this way eq. (5.8) can be related to the $\widehat{\mathrm{su}(m)}_1$ characters of the P-P sector.

[8]Let us recall that they are not characters of $\widehat{\mathrm{su}(m)}_1$ and then we explicitly break the su(m) symmetry.

For the twisted sector the separation into charged and neutral components is given by:[9]

$$\chi_{(s,\tilde{f})}(w|\tau) = \sum_{a=0}^{m-1} N^{(p)}_{(a,(s,\tilde{f}))}(\tau)\bar{K}^{[m(2pm+1)]}_{(2pm+1)a+s}(w|\tau) \tag{5.10}$$

where

$$N^{(p)}_{(a,(s,\tilde{f}))}(\tau) = \frac{1}{m}\sum_{j=0}^{m-1} e^{-\frac{2\pi i}{m}(2j)\left(\frac{\tilde{f}}{2}+\frac{s^2}{2(2pm+1)}-\frac{1}{24}-\frac{[(2pm+1)a+s]^2}{2(2pm+1)}\right)}\frac{\eta(\tau)}{\eta(\frac{\tau+2j}{m})}\,. \tag{5.11}$$

We must observe that the neutral part does not depend on p, that is on the flux attached to the charged component. That can be seen by showing that the characters of the neutral part, defined in the above formula, can be written in terms of the ones with $p = 0$ as: $N^{(p)}_{(a,(s,\tilde{f}))}(\tau) = N^{(0)}_{(a,\tilde{f}')}(\tau)$, where $\tilde{f}' = \tilde{f} - 2as \; mod \; (m)$.

Looking at eqs. (3.7), (5.8) and (5.10) and taking into account the parametrization change, we conclude that the charged component is the same in any sector of the TM and it coincides with the extended theory $\widehat{\mathrm{u}(1)}_{m(2mp+1)}$.

Let us notice that the characters given here are written in a basis which is explicitly invariant under the permutation of the Landau levels. In this basis we do not see the excitations content of the theory. We could introduce another basis in which the m-ality is diagonal, evidencing the primary fields with conformal dimensions $h_{(l,b)} = \frac{((2pm+1)l+mb)^2}{2m(2pm+1)} + \frac{l(m-l)}{m}$ which are the charged excitations corresponding to the l-electrons ($b = 0$) and the l-anions ($b \neq 0$) which are fixed under the twist group. Moreover in the twisted sector there are also neutral excitations with conformal dimensions $\bar{h} = \frac{m^2-1}{24m}$ (i.e. twist operators) which are a characteristics of the TM and it was not present in the models given in [3, 2]. They were independently introduced in [1] and [16] giving rise to a clustering phenomenon in the correlator of N electrons (see [1] for details). The role of the twisted sector has been recently clarified for the paired states in [29]. In that description the neutral current was shown to originate from a point-like interaction between the layers, which can be attributed to the presence of a localized impurity. We will give a similar analysis in a forthcoming paper.

5.2 The TM for $m = 2$

For $m = 2$ the TM is built following the same construction made in 5.1, for the case $m > 2$ and prime.

The content of the twisted sector is given by the m-reduction procedure applied to the mother theory with symmetry $\widehat{\mathrm{u}(1)}_{(4p+1)}$. The extended primary fields have the conformal dimensions

$$h_{(s,\tilde{f})} = \frac{s^2}{4\,(4p+1)} + \frac{1}{16} + \frac{\tilde{f}}{4} \tag{5.12}$$

with $s = 0, \ldots, 4p$, $\tilde{f} = 0, 1$, and the corresponding characters are given by

$$\chi_{(s,\tilde{f})}(w|\tau) = K^{(2(4p+1))}_{2((4p+1)\tilde{f}+s)}\left(w|\frac{\tau}{2}\right) \tag{5.13}$$

where $K^{(2(4p+1))}_q(w|\tau)$ are the characters of the ordinary[10] RCFT $\widehat{\mathrm{u}(1)}_{2(4p+1)}$.

[9]Notice that the m-ality charge in the twisted sector is $l = a + s \; mod \; (m)$.

[10]See appendix B.

The theory $\widehat{\mathrm{u}(1)}_K^{\otimes 2}$ is given by the P-P sector of the TM. The remaining part of its untwisted sector, that is the P-A part, is generated by the request of closure under $\Gamma_0(2)$. Thus the extended primary fields of the P-A part are obtained by performing the S modular transformation on the characters of the extended primary fields of the twisted sector. They have the following conformal dimension

$$\tilde{h}_{(\mu,g)} = \frac{((4p+1)\,g+\mu)^2}{4(4p+1)} \tag{5.14}$$

where $\mu = 0, \dots, 4p$, $g = 0, 1$, with correspondent characters

$$\tilde{\chi}_{(\mu,g)}(w|\tau) = \frac{1}{\sqrt{2}}\left(K^{(2(4p+1))}_{(4p+1)g+\mu}(2w|2\tau) + K^{(2(4p+1))}_{2(4p+1)+(4p+1)g+\mu}(2w|2\tau)\right). \tag{5.15}$$

We give now their separation into charged and neutral components and relate it to known results in the c=1 orbifold theory literature [27].

The characters of the twisted sector can be written as:

$$\chi_{(s,\tilde{f})}(w|\tau) = \chi_0^{\widehat{\mathrm{su}(2)}_1}\left(\frac{\rho\tau}{2}|\tau\right)\bar{K}^{(2(4p+1))}_{(4p+1)\tilde{f}+s}(w|\tau)\,, \tag{5.16}$$

while for the P-P sector eq. (3.7) gives the following separation:

$$\tilde{\chi}_b^{\mathbf{MC}}(w|\tau) = \chi_0^{\widehat{\mathrm{su}(2)}_1}(\tau)\bar{K}^{(2(4p+1))}_{2b}(w|\tau) + \chi_1^{\widehat{\mathrm{su}(2)}_1}(\tau)\bar{K}^{(2(4p+1))}_{(4p+1)+2b}(w|\tau) \tag{5.17}$$

$b = 0, \dots, 4p$. For the remaining P-A sector we get

$$\tilde{\chi}_{(\mu,g)}(w|\tau) = \chi_0^{\widehat{\mathrm{su}(2)}_1}\left(\frac{\rho}{2}|\tau\right)\bar{K}^{(2(4p+1))}_{(4p+1)g+\mu}(w|\tau) \tag{5.18}$$

where

$$\chi_0^{\widehat{\mathrm{su}(2)}_1}\left(\frac{\rho}{2}|\tau\right) = \frac{\Theta_4(0|2\tau)}{\eta}\,. \tag{5.19}$$

Eqs. (5.17), (5.18) and (5.16) show that the charged component of the TM for $m = 2$ is the extended theory $\widehat{\mathrm{u}(1)}_{2(4p+1)}$ which is an ordinary RCFT, closed under the full modular group. It justifies the particular modular behavior of the characters of the P-A sector which is closed under the modular transformation T. It is due to the bosonic nature of the 2-electron. It is worth noticing that even in such a case the full TM theory remains a $\Gamma_0(2)$ RCFT.

Finally we notice that the P-A sector is bigger than the theory built by the action of the generator of the discrete symmetry group Z_2. This can be seen by writing the results of section 5.1 for $m = 2$. The comparison of eq.(5.8), evaluated at $m = 2$, i.e.

$$\tilde{\chi}_b(w|\tau) = \chi_0^{\widehat{\mathrm{su}(2)}_1}\left(\frac{\rho}{2}|\tau\right)\bar{K}^{(2(4p+1))}_{2b}(w|\tau) \qquad b = 0, \dots, 4p \tag{5.20}$$

with the correspondent of eq.(5.18), makes it clear. Indeed, eq.(5.20) describes exactly half of the characters given in eq.(5.18).

This anomalous behavior can be understood by noticing that the $c = 1$ Z_2 -orbifold has an extra symmetry due to a $h = 1$ current ($\cos\phi$). Its radius corresponds to the multicritical point in which orbifold and gaussian theory coincide. The gaussian formulation can be obtained by means of a non-linear transformation as it was given in ref. [28].

The field content of the neutral sector can be easily deduced by the well know spectrum of the $c = 1$ Z_2-orbifold at the radius $R^2 = 4$. They are: I $(h = 0)$, $\partial\phi$ $(h = 1)$, ψ $(h = 1/4)$, $\bar{\psi}$ $(h = 1/4)$, σ_i $(h = 1/16)$, τ_i $(h = 9/16)$, ψ and $\bar{\psi}$ are two kinds of $\widehat{\mathrm{su}(2)}_1$ spinons. They can be immediately related to the standard basis as follows: $\psi = \psi^{(1)}+\psi^{(2)}$; $\bar{\psi} = \psi^{(1)}-\psi^{(2)}$.

Differently from the paired states case the fields ψ and $\bar{\psi}$ are not independent so they do not realize two independent CFTs as it happens in that case. They instead realize Gentile parafermions. As it is well known, a Gentile parafermion gives a $\frac{m-1}{m}$ contribution to the central charge. We need to have m of such a parafermions to build a unitary CFT with integer charge $c = m - 1$. As a consequence their fusion rules are: $\psi\times\psi = \partial\phi$; $\psi\times\bar{\psi} = I$; $\bar{\psi}\times\bar{\psi} = \partial\phi$.

Due to the lack of zero modes in the twisted current the pseudospin $J_3^{\mathrm{su}(2)} = m = (N^{(1)} - N^{(2)})/2$ is zero (or $1/2$) so that the ground state has an equal number of $\psi^{(1)}$ and $\psi^{(2)}$ states for $m = 0$ or $N^{(1)} = N^{(2)} + 1$ for $m = 1/2$. While ψ increases the (m-ality) $l \to l + 1$, $\bar{\psi}$ acts only on the third component as $m \to m + 1/2$.

6. The TM partition function

The diagonal partition function of the part of the TM generated only by the m-reduction procedure, without taking care of the $m - 1$ degeneracy, is

$$Z^{(m)}(w|\tau) = Z^{(m)}_{\mathrm{untwist}}(w|\tau) + Z^{(m)}_{\mathrm{twist}}(w|\tau) \tag{6.1}$$

where[11]

$$Z^{(m)}_{\mathrm{untwist}}(w|\tau) = \sum_{b=0}^{2pm} |\tilde{\chi}_b(w|\tau)|^2\ ; \qquad Z^{(m)}_{\mathrm{twist}}(w|\tau) = \sum_{\tilde{f}=0}^{m-1}\sum_{s=0}^{2pm} \left|\chi_{(s,\tilde{f})}(w|\tau)\right|^2 . \tag{6.3}$$

The diagonal partition function of the TM, as a consequence of its definition as a Z_m-orbifold, is

$$Z_{TM}(w|\tau) = \frac{1}{|Z_m|}\sum_{\forall(\pi_1,\pi_2)\in Z_m\times Z_m} Z^{\widehat{\mathrm{u}(1)}_K^{\otimes m}}_{(\pi_1,\pi_2)}(w|\tau) = \frac{1}{m}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} Z^{\widehat{\mathrm{u}(1)}_K^{\otimes m}}_{(g^i,g^j)}(w|\tau) \tag{6.4}$$

where we have defined

$$Z^{\widehat{\mathrm{u}(1)}_K^{\otimes m}}_{(g^i,g^j)}(w|\tau) = \sum_{s=0}^{2pm} \left|\tilde{\chi}_{(s,(g^i,g^j))}(w|\tau)\right|^2 \tag{6.5}$$

[11] In the $m = 2$ case all is the same with the only exception that

$$Z^{(2)}_{\mathrm{untwist}}(w|\tau) = \sum_{\mu=0}^{4p}\sum_{g=0}^{1} \left|\tilde{\chi}_{(\mu,g)}(w|\tau)\right|^2 \tag{6.2}$$

$i \in \{0, \ldots, m-1\}$, $j \in \{0, \ldots, m-1\}$. In particular the diagonal partition function of the P-P sector coincides with that of the $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ theory

$$Z_{(g^0,g^0)}^{\widehat{\mathrm{u}(1)}_K}(w|\tau) = \sum_{b=0}^{2pm} \left|\tilde{\chi}_b^{\mathbf{MC}}(w|\tau)\right|^2 . \tag{6.6}$$

Furthermore following the equalities:

$$Z_{\text{untwist}}^{(m)}(w|\tau) = \frac{1}{m} Z_{(\mathbf{1},g^i)}^{\widehat{\mathrm{u}(1)}_K^{\otimes m}}(w|\tau) \, ; \qquad Z_{\text{twist}}^{(m)}(w|\tau) = \frac{1}{m} \sum_{h=0}^{m-1} Z_{(g^i,g^h)}^{\widehat{\mathrm{u}(1)}_K^{\otimes m}}(w|\tau) \tag{6.7}$$

the TM partition function can be finally rewritten as:

$$Z_{TM}(w|\tau) = \frac{1}{m} Z_{\widehat{\mathrm{u}(1)}_K^{\otimes m}}(w|\tau) + (m-1)\, Z^{(m)}(w|\tau) \, . \tag{6.8}$$

7. The symmetries of the TM sectors

In this section we will discuss the relationship between the TM and the $\widehat{\mathrm{u}(1)}_{m(2pm+1)} \times \mathcal{W}_m$ minimal models in detail.

As we have already seen the minimal model is not a RCFT. The fact that the TM is the Z_m-orbifold of the $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ model clarifies the relation between the two previous proposed theories for the Jain plateaux: the multi-component bosonic theory, characterized by the extended $\widehat{\mathrm{u}(1)}_{m(2pm+1)} \times \widehat{\mathrm{su}(m)}_1$ algebra and the extended minimal model.

It is very easy to see that the $\widehat{\mathrm{u}(1)}_{m(2pm+1)} \times \widehat{\mathrm{su}(m)}_1$ algebra is the symmetry of the P-P sector of the TM. It is a non-minimal sector with respect to the $\mathcal{W}_m$ symmetry. To see that we can write, in the particular $m = 2$ case, eq. (3.12) which gives the characters of the affine $\widehat{\mathrm{su}(2)}_1$ in terms of the $\mathcal{W}_2$ characters:

$$\chi_l^{\widehat{\mathrm{su}(2)}_1}(\tau) = \sum_{n=0}^{\infty} d_{\mathrm{su}(2)} \left(n + \frac{l}{2}\right) \chi_{2n+l}^{\mathcal{W}_2}(\tau) \tag{7.1}$$

where $l = 0, 1$. They correspond to the fundamental weights $\hat{\mathbf{\Lambda}}_0, \hat{\mathbf{\Lambda}}_1$ of $\widehat{\mathrm{su}(2)}_1$ and can be written more explicitly as:

$$\chi_0^{\widehat{\mathrm{su}(2)}_1}(\tau) = \sum_{n=0}^{\infty} (2n+1)\, \chi_{2n}^{\mathcal{W}_2}(\tau) \; ; \; \chi_1^{\widehat{\mathrm{su}(2)}_1}(\tau) = \sum_{n=0}^{\infty} (2n+2)\, \chi_{2n+1}^{\mathcal{W}_2}(\tau) \, . \tag{7.2}$$

In the $m = 2$ case the algebra $\mathcal{W}_2$ has central charge $c = 1$ and its degenerate representations coincide with those of the Virasoro algebra. More precisely the primary fields of $\mathcal{W}_2$ with conformal dimension $L_0 = k^2/4$ have a null vector at level $n + 1$ of the relative representation for any $k \in N$. The characters of these Virasoro degenerate primary fields are then:

$$\chi_k^{\mathcal{W}_2}(\tau) = \frac{q^{\frac{k^2}{4}}(1 - q^{k+1})}{\eta(q)} \, . \tag{7.3}$$

Thus the $\hat{\mathbf{\Lambda}}_0, \hat{\mathbf{\Lambda}}_1$ h.w. representations of $\widehat{\mathrm{su}(2)}_1$ are built respectively by the even and odd degenerate representations of $\mathcal{W}_2$.

Now we will derive the identification of the neutral component of the P-A untwisted sector of our TM and show that it is related to the $\mathcal{W}_m$ minimal models.

There is just one extended primary field in the neutral component of the P-A sector of our TM and it can be written in terms of the $\mathcal{W}_2$ characters as:

$$\tilde{N}(\tau) = \chi_0^{\widehat{\mathrm{su}(2)}_1}\left(\frac{\rho}{2}|\tau\right) = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty} \chi_{2n}^{\mathrm{su}(2)}\left(\frac{\rho}{2}\right)\chi_{2n}^{\mathcal{W}_2}(\tau) \tag{7.4}$$

with $\chi_{2n}^{\mathrm{su}(2)}(\frac{\rho}{2}) = (-1)^n$. Eq. (7.4) tells us that the extended primary field of this sector is defined by collecting all the $2n$ degenerate primary fields of $\mathcal{W}_2$, everyone with multiplicity one times a residual parity symmetry $(-1)^n$. This extra parity can be seen as the residual symmetry of the finite su(2) algebra due to cocycles. As a consequence the P-A sector of the TM contains only degenerate representations of $\mathcal{W}_2$. Therefore we can consider it as a"minimal sector" of the full theory. Nevertheless in this sector the extended symmetry is given by $\mathrm{u}(1)\times\mathcal{W}_2\times Z_2$. We could take the Z_2 even superposition of $\mathcal{W}_2$ degenerate characters as it was done in [3] but we can see that only the odd superposition given by the characters $\tilde{\chi}_{(l,b)}$ of the TM belong to a RCFT. It is necessary to give a well defined S transformation for the characters of the P-A sector which produces the twisted A-P sector of the TM. In this sector the $\mathcal{W}_2$ characters are non minimal generic representations.

For generic m the extended characters in the P-A sector contains only $\mathcal{W}_m$ representations with $\lambda_a - \lambda_b \in mZ$ with the same structure of the $m = 2$ case. Thus only the weights in the $\hat{\mathbf{\Lambda}}_0$ representation contribute and furthermore, by the isomorphism $\widehat{\mathrm{su}(m)}_1 = \mathrm{su}(m)\times\mathcal{W}_m$, we have that in any su(m) multiplet just one is selected out. Instead in the A-P sector the differences $\lambda_a - \lambda_b \notin Z$ are not integer in general.

8. Conclusions

In this paper, by using the m-reduction procedure, we constructed the twisted model (TM) for the Jain series at the filling $\nu = \frac{m}{2pm+1}$, generalizing to the torus topology a recent construction for the plane [1]. The extended chiral primary fields were derived both for the twisted and untwisted sectors together with their corresponding characters. Also their factorization in terms of charged and neutral components was given. By constructing the complete diagonal partition function, which turns out to be invariant under $\Gamma_0(2)$ it has been shown that the multi-components theory coincides with its P-P sector content. A detailed relation between the TM, the multi-component and the $\mathcal{W}_m$ minimal model has been worked out throughout the paper. For $m = 2$ it has been explicitly shown that the P-A sector contains only degenerate representations of $\mathcal{W}_2$, so resulting into a "minimal sector" of the full theory. Finally it has been shown that the presence of the A-P sector assures the closure of the TM partition function under modular transformations. The special $m = 2$ case has been also analyzed. In the appendices it has been reported on the properties of the $\Gamma_0(2)$ group together with the modular transformations of the $\widehat{\mathrm{u}(1)}_q$ and TM characters.

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A. The $\Gamma_0(2)$ group

The $\Gamma_0(2)$ group is a subgroup of the modular group $\mathrm{SL}(2,\mathbb{Z})/Z_2$ generated by T^2 and T^2ST^2. A matrix representation of the generators T^2 and S of $\mathrm{SL}(2,\mathbb{Z})/Z_2$ is:

$$T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.1}$$

$\Gamma_0(2)$ is the group of elements defined as:

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})/Z_2 \ : \ a+d\,,\ b+c \text{ even}\,;\ a+b \text{ odd} \right\}. \tag{A.2}$$

Any element A of $\Gamma_0(2)$ can be represented as follows:

$$A = \begin{cases} T^{2a} & \forall a \in Z \\ S_{(a_1,b_1)} \times S_{(a_2,b_2)} \times \cdots \times S_{(a_N,b_N)} & \forall (a_j,b_j) \in Z \times Z\,,\ \forall j \in (1,\ldots,N) \\ & \forall N \in \mathbf{N} \end{cases} \tag{A.3}$$

where $S_{(a,b)} = T^{2a}ST^{2b}$. Thus the characterization given for the subgroup $\Gamma_0(2)$ is a direct consequence of the form of this matrix, in fact

$$T^{2a} = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix} \tag{A.4}$$

and

$$\prod_{j=1}^{N} S_{(a_j,b_j)} = \begin{cases} \begin{pmatrix} 2\alpha+1 & 2\beta \\ 2\gamma & 2\delta+1 \end{pmatrix} & \text{for } N \text{even} \\ \begin{pmatrix} 2\alpha & 2\beta+1 \\ 2\gamma+1 & 2\delta \end{pmatrix} & \text{for } N \text{odd} \end{cases} \tag{A.5}$$

where α, β, γ, δ are integers and depend on (a_j, b_j) and N.

B. The ECFT of the Laughlin filling

Here we want to construct the ECFT that describes the Laughlin filling $\nu = 1/q$ with $q = 2pm + 1$ odd.

This ECFT has to be a free bosonic theory compactified on a circle with radius $R^2 = q$, the topological order, that is the degeneracy of the quantum Hall ground state on the torus, has to coincide with the number of the extended primary fields of the ECFT [20]. Furthermore because the ground state is described by electrons (fermions), the behaviour required under modular transformations is not the closure under the entire modular group $\mathrm{SL}(2,\mathbb{Z})/Z_2$, but only under the subgroup $\Gamma_0(2)$.

The "smaller" RCFT of a free boson with $R^2 = q$(here it means $R^2 = 2q/2$ o by duality $R^2 = 4/q$) with q odd, is a $\widehat{\mathrm{u}(1)}_k$ of level $k = 2q$. The extended primary fields are defined by the vertex operator:

$$V_l^{(2q)}(z) =: e^{i\alpha_l \phi(z)} : \qquad \text{with } \alpha_l = \frac{l}{\sqrt{4q}}\,, \qquad l = 0, \ldots, 4q-1 \tag{B.1}$$

and the correspondent characters are:

$$K_l^{(2q)}(w|\tau) = \frac{e^{-q\pi\frac{(Imw)^2}{Im\tau}}}{\eta(\tau)}\Theta\begin{bmatrix}\frac{l}{4q}\\0\end{bmatrix}(2qw|4q\tau)\,. \tag{B.2}$$

There are $4q$ extended primary fields and the correspondent characters are closed under the entire modular group, $\mathrm{SL}(2,\mathbb{Z})/Z_2$. Thus the ordinary RCFT $\widehat{\mathrm{u}(1)}_{2q}$ does not satisfy the requests of an ECFT for the Laughlin filling $\nu = 1/q$.

The ECFT which does is not an ordinary RCFT. It can be built starting from $\widehat{\mathrm{u}(1)}_{2q}$, by defining in a different way the extended primary fields. The chiral extended algebra is defined by assembling in any extended primary field all the primary fields of $\widehat{\mathrm{u}(1)}_{2q}$, whose conformal dimensions differ by half integer. We define this extended theory as $\widehat{\mathrm{u}(1)}_q$. Its extended primary fields are defined by the vertex operators[12]

$$V_a^{(q)}(z) =: e^{i\alpha_a\phi(z)}: \qquad \text{with } \alpha_a = \frac{a}{\sqrt{q}} \tag{B.3}$$

where $a = 0, \dots, q-1$ and the correspondent characters are:

$$\bar{K}_a^{(q)}(w|\tau) = \frac{e^{-q\pi\frac{(Imw)^2}{Im\tau}}}{\eta(\tau)}\Theta\begin{bmatrix}\frac{a}{q}\\0\end{bmatrix}(qw|q\tau)\,. \tag{B.4}$$

There are q extended primary fields and the relative characters are closed only under the subgroup $\Gamma_0(2)$. Thus the $\Gamma_0(2)$ RCFT $\widehat{\mathrm{u}(1)}_q$ satisfies the requests of an ECFT for the Laughlin filling, $\nu = 1/q$.

Finally we notice that the extended theory $\widehat{\mathrm{u}(1)}_q$ coincides with the $\Gamma_0(2)$-invariant part of the ordinary RCFT $\widehat{\mathrm{u}(1)}_{2q}$. This is a consequence of the relation among the characters of the two theories:

$$\bar{K}_a^{(q)}(w|\tau) = K_{2a}^{(2q)}(w|\tau) + K_{2(q+a)}^{(2q)}(w|\tau) \qquad \forall a \in \{1, \dots, q\}\,. \tag{B.5}$$

The modular transformations for the $\widehat{\mathrm{u}(1)}_q$ characters. The modular transformations for the $\widehat{\mathrm{u}(1)}_q$ characters are derived by the well known modular transformations for the characters of the ordinary RCFT $\widehat{\mathrm{u}(1)}_{2q}$ appearing in eq. (B.5).

The transformation T^2 acts as:

$$\bar{K}_s^{(q)}(w|\tau+2) = e^{i4\pi\left(\frac{s^2}{2q}-\frac{1}{24}\right)}\bar{K}_s^{(q)}(w|\tau)\,. \tag{B.6}$$

The transformation S acts as:

$$\bar{K}_s^{(q)}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) = \frac{e^{iq\pi Re\frac{w^2}{\tau}}}{\sqrt{q}}\sum_{s'=0}^{q-1} e^{\frac{2i\pi s's}{q}}\bar{K}_{s'}^{(q)}(w|\tau)\,. \tag{B.7}$$

[12] These are just the primary fields representative of the entire conformal blocks that define the extended primary fields.

C. The modular transformations of the TM characters

C.1 The case $m > 2$ and prime

The modular transformations of the characters of $\widehat{\mathrm{u}(1)}_K^{\otimes m}$. As we have seen the theory $\widehat{\mathrm{u}(1)}_K^{\otimes m}$ coincides with the P-P sector of TM. Thus we write the modular transformation of the relative characters.

Under T^2 we get:

$$\tilde{\chi}_b^{\mathbf{MC}}(w|\tau+2) = e^{i4\pi\left(\frac{mb^2}{2(2pm+1)}-\frac{m}{24}\right)}\tilde{\chi}_b^{\mathbf{MC}}(w|\tau)\,. \tag{C.1}$$

Under S we get:

$$\tilde{\chi}_b^{\mathbf{MC}}(\frac{w}{\tau}|-\frac{1}{\tau}) = \frac{e^{im(2pm+1)\pi Re\frac{w^2}{\tau}}}{\sqrt{2pm+1}}\sum_{b'=0}^{2pm} e^{\frac{2i\pi mbb'}{2pm+1}}\tilde{\chi}_{b'}^{\mathbf{MC}}(w|\tau)\,. \tag{C.2}$$

The modular transformations of the characters built by the m-reduction procedure. Here we write the modular transformations for the TM characters $\chi_{(s,\tilde{f})}(w|\tau)$, $\tilde{\chi}_b(w|\tau)$.

Under T^2 we obtain:

$$\chi_{(s,\tilde{f})}(w|\tau+2) = e^{\frac{4\pi i}{m}(\frac{s^2}{2(2pm+1)}+\frac{\tilde{f}}{2}-\frac{1}{24})}\chi_{(s,\tilde{f})}(w|\tau) \tag{C.3}$$

$$\tilde{\chi}_b(w|\tau+2) = e^{i4\pi\left(\frac{mb^2}{2(2pm+1)}-\frac{m}{24}\right)}\tilde{\chi}_b(w|\tau)\,. \tag{C.4}$$

Under S it is:

$$\begin{aligned}\chi_{(s,\tilde{f})}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) &= e^{i\pi m(2pm+1)Re\frac{w^2}{\tau}}\sum_{s'=0}^{2pm}\sum_{f=0}^{m-1}\mathcal{S}_{(s,\tilde{f})}^{(s',f)}\chi_{(s',f)}(w|\tau) + \\ &\quad+\frac{e^{i\pi m(2pm+1)Re\frac{w^2}{\tau}}}{\sqrt{m\,(2pm+1)}}\sum_{b=0}^{2pm}e^{\frac{2i\pi sb}{2pm+1}}\tilde{\chi}_b(w|\tau)\end{aligned} \tag{C.5}$$

$$\tilde{\chi}_b\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) = \frac{e^{i\pi m(2pm+1)Re\frac{w^2}{\tau}}}{\sqrt{m\,(2pm+1)}}\sum_{s=0}^{2pm}\sum_{f=0}^{m-1}e^{\frac{2i\pi bs}{2pm+1}}\chi_{(s,f)}(w|\tau) \tag{C.6}$$

with

$$\mathcal{S}_{(s,\tilde{f})}^{(s',f)} = \frac{1}{m}\sum_{j=1}^{m-1}\left(e^{\frac{2\pi i}{m}(2j^*)(\frac{s'^2}{2(2pm+1)}+\frac{f}{2}-\frac{1}{24})}\left(\mathcal{A}_{(m,p,2j)}\right)_{(s,s')}e^{-\frac{2\pi i}{m}(2j)(\frac{s^2}{2(2pm+1)}+\frac{\tilde{f}}{2}-\frac{1}{24})}\right)$$

where $(\mathcal{A}_{(m,p,2j)})_{(s,s')}$ are the entries of the $(2pm+1)\times(2pm+1)$ matrix $\mathcal{A}_{(m,p,2j)}$ that represent the action of the modular transformation $A_{(m,2j)} \in \Gamma_0(2)$ on the characters of the $\widehat{\mathrm{u}(1)}_{2pm+1}$ theory. The matrix $A_{(m,2j)} \in \Gamma_0(2)$ and the integer number j^* are defined in a univocal way by the conditions:

$$\frac{(-1/\tau)+2j}{m} = A_{(m,2j)}\left(\frac{\tau+2j^*}{m}\right) \qquad \forall j \in (1,\ldots,m-1)\,. \tag{C.7}$$

In fact eq. (C.7) is satisfied only for:

$$A_{(m,2j)} = \begin{pmatrix} 2j & b-4j\alpha \\ m & 2j^* \end{pmatrix}, \qquad j^* = \alpha m - j' \tag{C.8}$$

with j' a nonzero integer number with minimal modulo, and b an odd integer number, defined by the relation:

$$(2j)\left(2j'\right) - bm = 1\,. \tag{C.9}$$

For $m > 2$ and prime and for $j = 1, \ldots, m-1$, eq.(C.9) simply expresses that m and $2j$ are coprime numbers. Finally the value of α is uniquely defined by the request $j^* = 1, \ldots, m-1$.

The fact that the matrices $A_{(m,2j)}$ are elements of $\Gamma_0(2)$ is a direct consequence of the characterization given in appendix A. By using eq.(A.5) this matrix can be expressed in terms of the matrices T^2 and S by:

$$A_{(m,2j)} = S_{(a_1,b_1)} \times S_{(a_2,b_2)} \times \cdots \times S_{(a_N,b_N)} \tag{C.10}$$

where $S_{(a,b)} = T^{2a}ST^{2b}$, are 2×2 matrices, N is an odd positive integer, $(a_j, b_j) \in Z\times Z$, for $j = 1, \ldots, N$.

It is important to point out that the condition $A_{(m,2j)} \in \Gamma_0(2)$ is crucial to obtain the closure under the modular transformation S of the characters $\chi_{(s,\tilde{f})}(w|\tau)$, $\tilde{\chi}_b(w|\tau)$.

C.2 The case $m = 2$

The modular transformation of the characters of the P-P untwisted sector are simply obtained evaluating at $m = 2$ eqs. (C.1) and (C.2).

The modular transformations of the remaining characters are given by ($q = 4p + 1$):

$$T^2: \ \chi_{(s,\tilde{f})}(w|\tau+2) = e^{2\pi i(\frac{s^2}{2q} + \frac{\tilde{f}}{2} - \frac{1}{24})} \chi_{(s,\tilde{f})}(w|\tau) \tag{C.11}$$

$$S: \ \chi_{(s,\tilde{f})}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) = \frac{e^{2\pi i q Re \frac{w^2}{\tau}}}{\sqrt{2q}} \sum_{\mu=0}^{q-1} \sum_{g=0}^{1} e^{\frac{2\pi i}{2q}(s+q\tilde{f})(\mu+qg)} \tilde{\chi}_{(\mu,g)}(w|\tau) \tag{C.12}$$

$$T: \ \tilde{\chi}_{(\mu,g)}(w|\tau+1) = e^{2\pi i\left(\frac{(\mu+qg)^2}{q} - \frac{1}{12}\right)} \tilde{\chi}_{(\mu,g)}(w|\tau) \tag{C.13}$$

$$S: \ \tilde{\chi}_{(\mu,g)}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) = \frac{e^{2\pi i q Re \frac{w^2}{\tau}}}{\sqrt{2q}} \sum_{s=0}^{q-1} \sum_{\tilde{f}=0}^{1} e^{\frac{2\pi i}{2q}(s+q\tilde{f})(\mu+qg)} \chi_{(s,\tilde{f})}(w|\tau)\,. \tag{C.14}$$

Eq. (C.13) shows that for $m = 2$ the characters $\tilde{\chi}_{(\mu,g)}$ are closed under T. While in the case $m > 2$ and prime number all the characters of TM are closed only under T^2; however in any case the TM is a $\Gamma_0(2)$-RCFT.

The closure under $\Gamma_0(2)$ of the diagonal partition function of TM. It is an immediate consequence of the given modular transformations of the characters of the TM.

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In order to show more simply the effect of the modular transformations we define:[13]

$$\begin{aligned} Z^{(m)+}_{\rm twist}(w|\tau) &= \frac{1}{m}\sum_{s=0}^{2pm}\left|\bar{K}_s^{(1+2pm)}\left(w|\frac{\tau}{m}\right)\right|^2 \\ Z^{(m)-}_{\rm twist}(w|\tau) &= \frac{1}{m}\sum_{h=1}^{m-1}\sum_{s=0}^{2pm}\left|\bar{K}_s^{(1+2pm)}\left(w|\frac{\tau+2h}{m}\right)\right|^2 . \end{aligned} \tag{C.15}$$

In terms of them $Z^{(m)}_{\rm twist}(w|\tau)$ is written as:

$$Z^{(m)}_{\rm twist}(w|\tau) = Z^{(m)+}_{\rm twist}(w|\tau) + Z^{(m)-}_{\rm twist}(w|\tau)\,. \tag{C.16}$$

The modular transformation T^2:

$$\begin{aligned} Z_{\widehat{{\rm u}(1)}_K^{\otimes m}}(w|\tau+2) &= Z_{\widehat{{\rm u}(1)}_K^{\otimes m}}(w|\tau)\,; \qquad Z^{(m)}_{\rm twist}(w|\tau+2) = Z^{(m)}_{\rm twist}(w|\tau) \\ Z^{(m)}_{\rm untwist}(w|\tau+2) &= Z^{(m)}_{\rm untwist}(w|\tau)\,. \end{aligned} \tag{C.17}$$

The modular transformation S:

$$\begin{aligned} Z_{\widehat{{\rm u}(1)}_K^{\otimes m}}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) &= Z_{\widehat{{\rm u}(1)}_K^{\otimes m}}(w|\tau)\,; \qquad Z^{(m)}_{\rm untwist}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) = Z^{(m)+}_{\rm twist}(w|\tau) \\ Z^{(m)+}_{\rm twist}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) &= Z^{(m)}_{\rm untwist}(w|\tau)\,; \qquad Z^{(m)-}_{\rm twist}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) = Z^{(m)-}_{\rm twist}(w|\tau)\,. \end{aligned} \tag{C.18}$$

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[13]For $m=2$ we have not the terms $Z^{(2)+}_{\rm twist}(w|\tau)$, $Z^{(2)-}_{\rm twist}(w|\tau)$ and the S transformation becomes

$$\begin{aligned} Z_{{\rm u}(1)_K^{\otimes 2}}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) &= Z_{{\rm u}(1)_K^{\otimes 2}}(w|\tau)\,; \qquad Z^{(2)}_{\rm untwist}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) = Z^{(2)}_{\rm twist}(w|\tau) \\ Z^{(2)}_{\rm twist}\left(\frac{w}{\tau}|-\frac{1}{\tau}\right) &= Z^{(2)}_{\rm untwist}(w|\tau)\,. \end{aligned}$$

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